Independence in abstract elementary classes

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- We provide the following answer in the framework of abstract elementary classes (AECs):

Theorem

Let K be a fully tame and short AEC with a monster model. Assume K is categorical in unboundedly many cardinals.

Then there exists λ such that $K_{\geq \lambda}$ admits an independence notion with all the properties of forking in a superstable first-order theory (except it may only have extension over saturated models).

Abstract elementary classes

Definition (Shelah, 1985)

Let K be a nonempty class of structures of the same similarity type $L(K)$, and let $\leq$ be a partial order on K. $(K, \leq)$ is an *abstract elementary class (AEC)* if it satisfies:

1. K is closed under isomorphism, $\leq$ respects isomorphisms.
2. If $M \leq N$ are in K, then $M \subseteq N$.
3. Coherence: If $M_0 \subseteq M_1 \leq M_2$ are in K and $M_0 \leq M_2$, then $M_0 \leq M_1$.
4. Downward Löwenheim-Skolem axiom: There is a cardinal $\mathrm{LS}(K) \geq |L(K)| + \aleph_0$ such that for any $N \in K$ and $A \subseteq |N|$, there exists $M \leq N$ containing A of size $\leq \mathrm{LS}(K) + |A|$.
5. Chain axioms: If δ is a limit ordinal, $\langle M_i : i < \delta \rangle$ is a $\leq$-increasing chain in K, then $M := \bigcup_{i<\delta} M_i$ is in K, and:
   5.1 $M_0 \leq M$.
   5.2 If $N \in K$ is such that $M_i \leq N$ for all $i < \delta$, then $M \leq N$.

Example of an AEC

For $\psi \in L_{\omega_1,\omega}$, Φ a countable fragment containing ψ,
$K := (\mathrm{Mod}(\psi), \prec_\Phi)$ is an AEC with $\mathrm{LS}(K) = \aleph_0$.

Two approaches to AECs

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Question (The global approach to AECs)

Work in ZFC, but make *global* model-theoretic hypotheses (like a monster model or locality conditions on types). What can we say about the AEC?

Global assumptions

Throughout the talk, we fix an AEC K. We assume we work inside a "big" model-homogeneous universal model $\mathfrak{C}$.

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Definition (Galois types)

For $\bar{b} \in {}^{<\infty}\mathfrak{C}$, $A \subseteq |\mathfrak{C}|$, let $\text{gtp}(\bar{b}/A)$ be the orbit of $\bar{b}$ under the automorphisms of $\mathfrak{C}$ fixing A.

Tameness

Let κ be an infinite cardinal.

Definition (Grossberg-VanDieren, 2006)

K is $(<\kappa)$-*tame* if for any M and any *distinct* $p, q \in \mathrm{gS}(M)$, there exists $A \subseteq |M|$ of size less than κ such that $p \upharpoonright A \neq q \upharpoonright A$.

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Definition (Boney, 2013)

K is *fully* $(< \kappa)$-*tame and short* if for any α, any M, and any distinct $p, q \in \text{gS}^\alpha(M)$, there exists $A \subseteq |M|$ and $I \subseteq \alpha$ of size less than κ such that $p^I \upharpoonright A \neq q^I \upharpoonright A$.

Tame AECs and large cardinals

Fact (Makkai-Shelah, Boney)

Let $\kappa > \text{LS}(K)$ be strongly compact. Then:

1. (No need for K to have a monster model) If K is categorical in some $\lambda > \beth_{\kappa+1}(\kappa)$, then $K_{\geq \kappa}$ has a monster model.

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2. K is fully $(< \kappa)$-tame and short.

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   - 2.2 **Monotonicity**: if $M \leq M' \leq N' \leq N$, $I \subseteq \alpha$, and $p \in \mathrm{gS}^{\alpha}(N)$ dnf over M, then $p^I \upharpoonright N'$ dnf over M'.

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   2.3 **Existence of unique extension**: If $p \in \mathrm{gS}^{\alpha}(M)$ and $N \geq M$, there exists a unique $q \in \mathrm{gS}^{\alpha}(N)$ extending p and not forking over M. Moreover q is algebraic if and only if p is.

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   2.4 **Set local character**: If $p \in \mathrm{gS}^{\alpha}(M)$, there exists $M_0 \leq M$ with $\|M_0\| \leq |\alpha| + \mathrm{LS}(K)$ such that p dnf over M_0.

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   - 2.5 **Chain local character**: If $\langle M_i : i \leq \delta \rangle$ is increasing continuous, $p \in \mathrm{gS}^{\alpha}(M_\delta)$ and $\mathrm{cf}(\delta) > \alpha$, then there exists $i < \delta$ such that p dnf over M_i.

Localizing goodness

- For α a cardinal, $\mathcal{F}$ an interval of cardinals, we say K is $(< \alpha, \mathcal{F})$-*good* if it is good when we restrict types to have length less than α, and models to have size in $\mathcal{F}$.

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- For α a cardinal, $\mathcal{F}$ an interval of cardinals, we say K is $(< \alpha, \mathcal{F})$-*good* if it is good when we restrict types to have length less than α, and models to have size in $\mathcal{F}$.
- For example, good means $(< \infty, \geq \text{LS}(K))$-good. In Shelah's terminology, $(\leq 1, \geq \lambda)$-good means K has a type-full good $(\geq \lambda)$-frame.

Challenges in proving goodness

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- **A key question**: If $\langle p_i : i \leq \delta \rangle$ is an increasing continuous chain of types and each p_i does not fork over M_0 for $i < \delta$, do we have that p_δ does not fork over M_0?
- For types of length one, this follows from local character.
- But for infinite types, this is much harder.

Some previous work on independence in AECs

Fact (Shelah)

Let K be an AEC, categorical in λ, λ^+, with at least one but "few" models in λ^{++}.

If $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and the weak diamond ideal on λ^+ is not λ^{++}-saturated, then K is $(\leq \lambda^+, \lambda^+)$-good.

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Fact (V.)

If K is $(\leq \mu)$-tame and categorical in a λ with $\text{cf}(\lambda) > \mu$, then K is $(\leq 1, \geq \lambda)$-good.

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Let $\kappa > \text{LS}(K)$ be strongly compact and let K be categorical in a $\lambda = \lambda^{<\kappa}$. Then $K_{\geq \lambda}$ is good.

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1. If K is $(< \kappa)$-tame, then $K_{\geq \lambda}$ is $(\leq 1, \geq \lambda)$-good.
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Corollary

If K is $(<\kappa)$-tame, $\kappa = \beth_\kappa > \mathrm{LS}(K)$, and K is categorical in a $\lambda > \kappa$, then K is stable in *all* cardinals.

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Remark

We can replace categoricity by a natural definition of superstability, analog to $\kappa(T) = \aleph_0$.

Shelah's categoricity conjecture from large cardinals?

Conjecture (Shelah)

Let K be an AEC. If K is categorical in unboundedly many cardinals, then K is categorical on a tail of cardinals.

[1] Shelah claims stronger results in Chapter IV of his book.

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Claim (Shelah, to appear in Sh:842)

If K has an ω-successful good λ-frame and weak GCH holds, then K is categorical in *some* $\mu > \lambda^{+\omega}$ if and only if K is categorical in *all* $\mu > \lambda^{+\omega}$.

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It turns out our construction gives an ω-successful good frame. Thus *modulo Shelah's claim*, we get[1]:

Corollary

Assume weak GCH. If there are unboundedly many strongly compact cardinals, then Shelah's categoricity conjecture holds.

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Main steps of the proof

Fix a "nice-enough" AEC K.

1. Using methods such as Galois-Morleyization and previous results of Boney-Grossberg, show that coheir has some (not all) of the properties of a good independence relation.

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4. Use a strong continuity property proven by Shelah as well as tameness and shortness to obtain a good $(\leq \lambda, \geq \lambda)$-independence relation.
5. Use tameness and shortness to obtain a good $(< \infty, \geq \lambda)$-independence relation (we can only prove extension over saturated models).

Thank you!

- For further reference, see:
  Sebastien Vasey, *Independence in abstract elementary classes*.
- A preprint can be accessed from my webpage:
  `http://svasey.org/`
- For a direct link, you can take a picture of the QR code below: